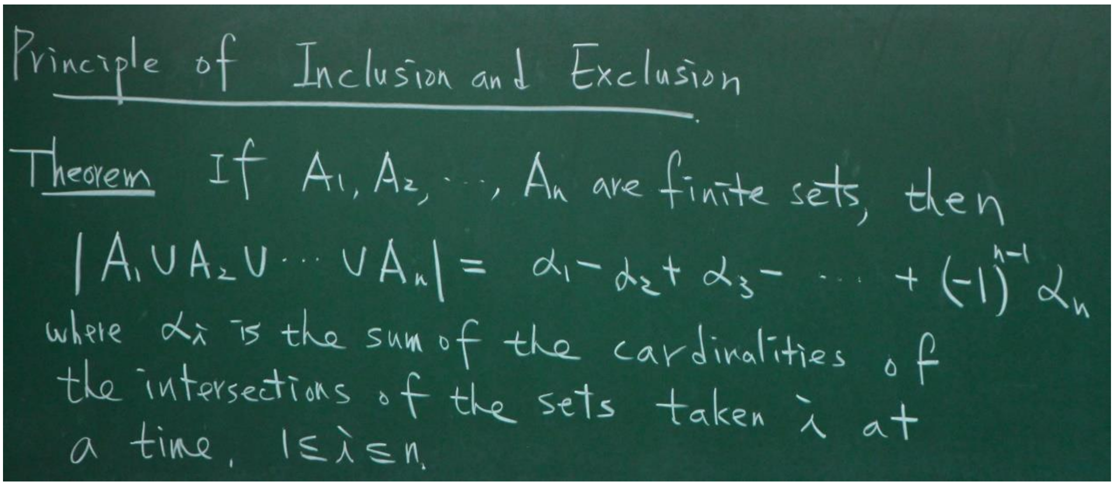
【10920 趙啟超教授離散數學 / 第 11 堂版書】

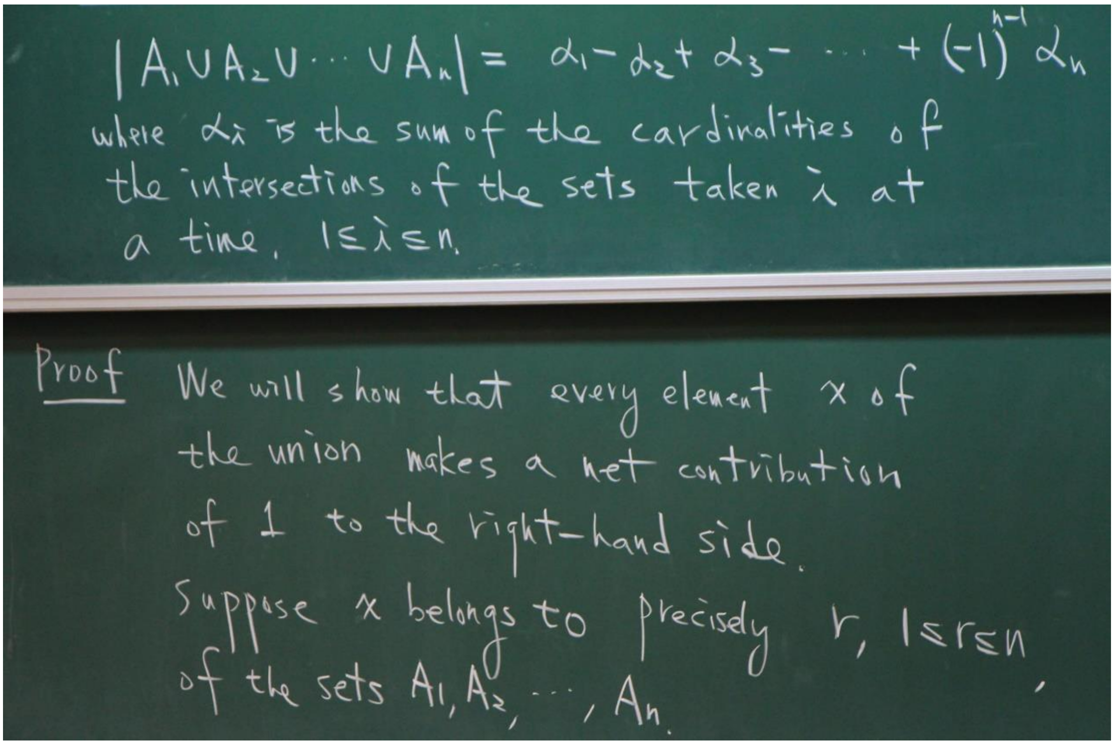


Principle of Inclusion and Exclusion

Theorem If $A_1, A_2, \cdots, A_n$ are finite sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \alpha_1 - \alpha_2 + \alpha_3 - \cdots + (-1)^{n-1}\alpha_n$$

where α_i is the sum of the cardinalities of the intersections of the sets taken i at a time, $1 \le i \le n$.



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Proof We will show that every element x of the union makes a net contribution of 1 to the right-hand side.

Suppose x belongs to precisely r, $1 \le r \le n$, of the sets $A_1, A_2, \cdots, A_n$.

In $\alpha_1 = |A_1| + |A_2| + \cdots + |A_n|$,

x contributes r to α_1.

In α_2, x contributes 1 to $|A_i \cap A_j|$ when both A_i and A_j are among the r sets which contain x. There are $\binom{r}{2}$ such pairs, and so $\binom{r}{2}$ is the contribution of x to α_2.

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In general, the contribution of x to α_i, $1 \le i \le r$, is $\binom{r}{i}$.

If $i > r$, the contribution of x to α_i is 0.

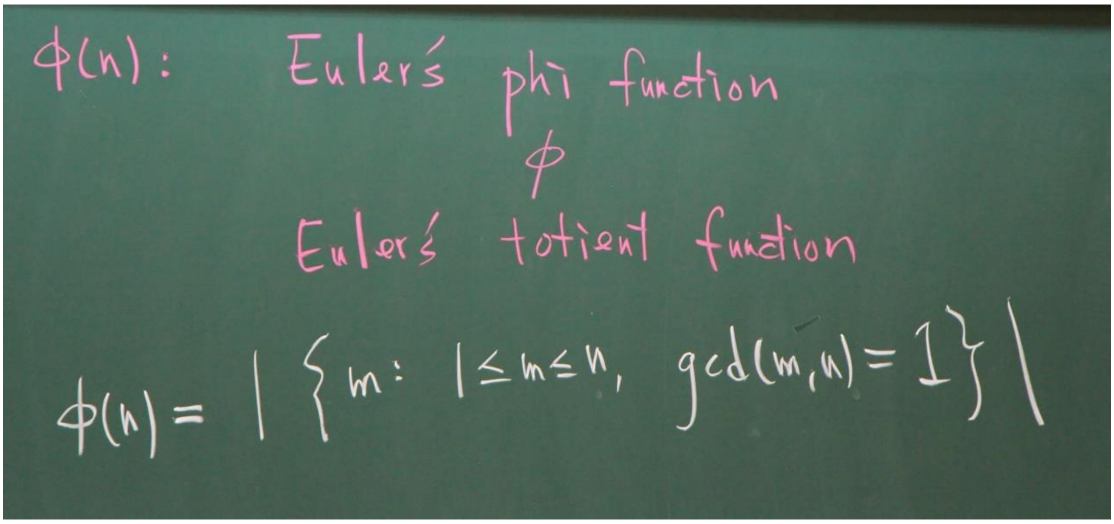
Hence the net contribution of x to the right-hand side is

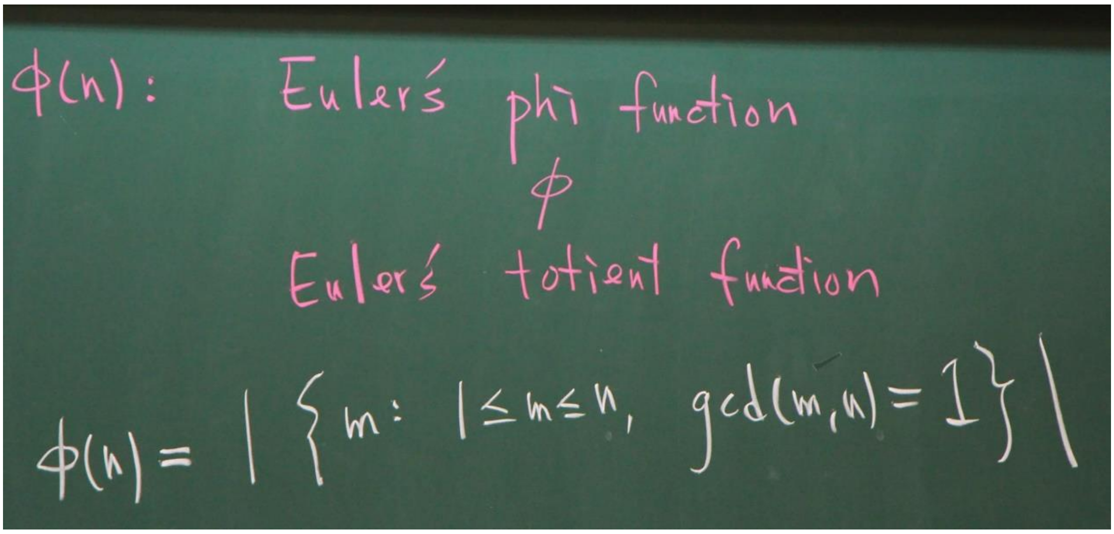
$$\binom{r}{1} - \binom{r}{2} + \cdots + (-1)^{i-1}\binom{r}{i} + \cdots + (-1)^{r-1}\binom{r}{r}$$

$$= \binom{r}{0} = 1$$

since $\binom{r}{0} - \binom{r}{1} + \binom{r}{2} + \cdots + (-1)^r \binom{r}{r} = 0$.

<u>Example</u> Let $\phi(n)$ denote the number of integers m in the range $1 \le m \le n$ such that m and n are relatively prime, i.e., $\gcd(m,n) = 1$.


$\phi(n)$: Euler's phi function
ϕ
Euler's totient function
$\phi(n) = |\{m : 1 \le m \le n,\ \gcd(m,n) = 1\}|$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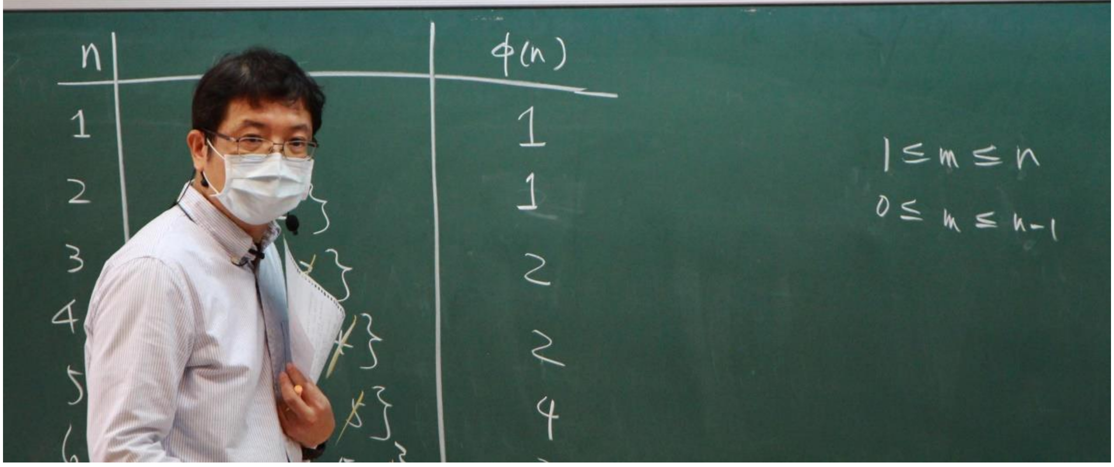

n		$\phi(n)$
1	$\{1\}$	1
2	$\{1, \cancel{2}\}$	1
3	$\{1, 2, \cancel{3}\}$	2
4	$\{1, \cancel{2}, 3, \cancel{4}\}$	2
5	$\{1, 2, 3, 4, \cancel{5}\}$	4
6	$\{1, \cancel{2}, \cancel{3}, \cancel{4}, 5, \cancel{6}\}$	2

$1 \le m \le n \qquad \gcd(m, n) = 1$

$0 \le m \le n-1 \qquad \gcd(m, n) = 1$

4	$\{1, \cancel{2}, 3, \cancel{4}\}$	2
5	$\{1, 2, 3, 4, \cancel{5}\}$	4
6	$\{1, \cancel{2}, \cancel{3}, \cancel{4}, 5, \cancel{6}\}$	2

$\phi(p) = p - 1$ if p is a prime

Find $\phi(60)$.

We have $60 = 2^2 \cdot 3 \cdot 5$

and hence $\phi(60) = \left|\{m : 1 \le m \le 60,\ \gcd(m, 60) = 1\}\right|$

$= \left|\{m : 1 \le m \le 60,\ 2 \nmid m,\ 3 \nmid m,\ 5 \nmid m\}\right|$

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Let $A_i = \{m : 1 \le m \le 60,\ i \mid m\}$, $i = 2, 3, 5$.

Then $\phi(60) = |\bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_5|$

$= |\overline{A_2 \cup A_3 \cup A_5}|$

$= 60 - |A_2 \cup A_3 \cup A_5|$

$$= \left| \overline{A_2 \cup A_3 \cup A_5} \right|$$

$$= 60 - |A_2 \cup A_3 \cup A_5|$$

$$= 60 - \big(|A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_3 \cap A_5| - |A_5 \cap A_2| + |A_2 \cap A_3 \cap A_5| \big)$$

$$= 60 - \left(\frac{60}{2} + \frac{60}{3} + \frac{60}{5} - \frac{60}{2 \cdot 3} - \frac{60}{3 \cdot 5} - \frac{60}{5 \cdot 2} + \frac{60}{2 \cdot 3 \cdot 5} \right)$$

$$= 60\left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 2} - \frac{1}{2 \cdot 3 \cdot 5} \right)$$

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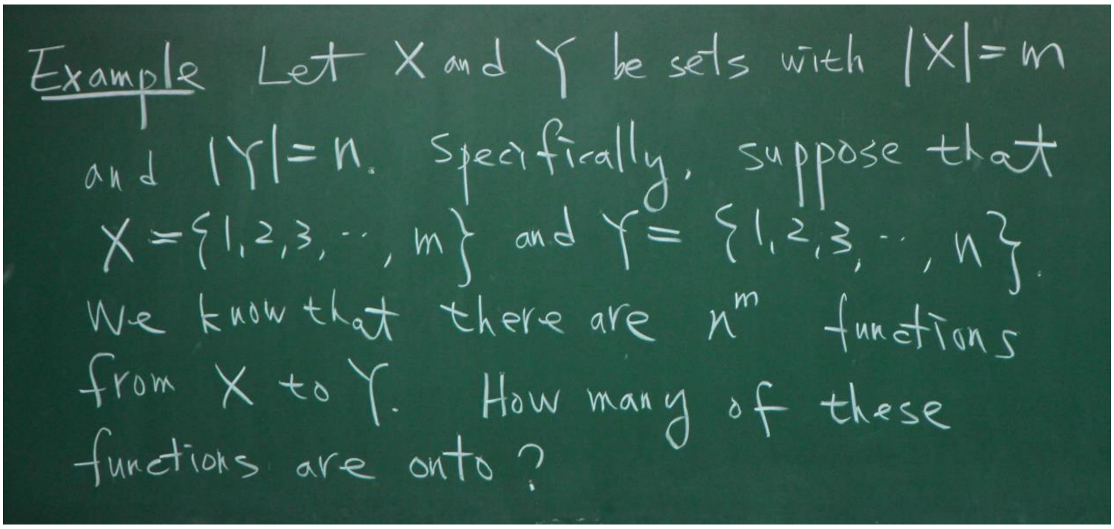
Example Let X and Y be sets with |X| = m
and |Y| = n. Specifically, suppose that
X = {1, 2, 3, ⋯, m} and Y = {1, 2, 3, ⋯, n}.
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Let this number be denoted by $\mathrm{onto}(m, n)$.

Clearly,

$$\mathrm{onto}(m, n) = 0 \quad \text{if } m < n.$$

$$\mathrm{onto}(m, m) = m! \quad \text{for all } m \geq 1$$

$$\mathrm{onto}(m, 1) = 1 \quad \text{for all } m \geq 1$$

$$\mathrm{onto}(m, 2) = 2^m - 2 \quad \text{for all } m \geq 2$$

Let A_i denote the set of functions from X to Y which do not take on the value i in Y, $1 \leq i \leq n$.

Then

$$\mathrm{onto}(m, n) = |\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n|$$

$$= |\overline{A_1 \cup A_2 \cup \cdots \cup A_n}|$$

$$= |S| - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

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(where S is the set of all functions from X to Y).

We have $|A_i| = (n-1)^m$

$|A_i \cap A_j| = (n-2)^m$ for $i \ne j$

$\vdots$

$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}| = (n-r)^m$ for distinct $i_1, i_2, \cdots, i_r$

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for $1 \le r \le n$.

Hence $\alpha_1 = |A_1| + |A_2| + \cdots + |A_n| = n \cdot (n-1)^m$

$\alpha_2 = |A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|$

$= \binom{n}{2} (n-2)^m$

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$\alpha_r = \binom{n}{r} (n-r)^m$ for $1 \le r \le n$

$\alpha_2 = |A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|$

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$\vdots$

$\alpha_r = \binom{n}{r} (n-r)^m$ for $1 \le r \le n$

$$\therefore \quad \operatorname{onto}(m,n) = n^m - \sum_{r=1}^{n} (-1)^{r-1} \binom{n}{r} (n-r)^m$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} (n-r)^m \qquad \text{for } m \ge n$$

Hence $\alpha_1 = |A_1| + |A_2| + \cdots + |A_n| = n \cdot (n-1)^m$

$$\alpha_2 = |A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|$$
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for $m \ge n$

For $m \ge n$, onto(m,n) is also the number of ways to distribute m distinct objects into n numbered (but otherwise identical) containers with no container left empty.

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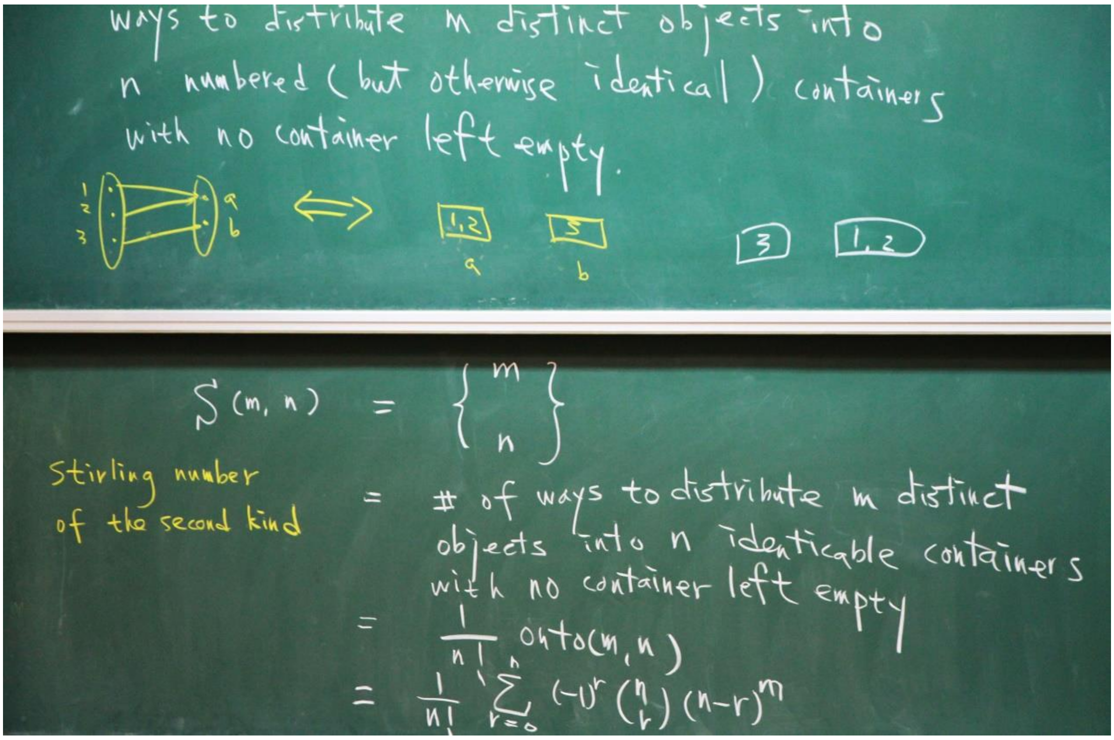
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1
2
3
a
b
1,2
a
3
b
3
1,2
$S(m, n) = \left\{ {m \atop n} \right\}$
Stirling number of the second kind
= # of ways to distribute m distinct objects into n identicable containers with no container left empty
$= \frac{1}{n!}\,\mathrm{onto}(m, n)$
$= \frac{1}{n!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} (n-r)^m$